B.Sc. part II paper IV

Differentiation of vectors functions

① Differentiation of vectors

Def:- Let t is a scalar variable and $\vec{r} = \vec{f}(t)$ be a vector function then differentiation of $\vec{r}$ with respect to t is denoted by $\frac{d\vec{r}}{dt}$ and defined as

$$\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{f(t+\delta t) - \vec{f}(t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t} \text{ if limit exists}$$

② partial derivatives of vectors:

Def:- Let $\vec{r}(x,y,z)$ is a continuous vector function then the partial derivatives of $\vec{r}$ w.r.t 'x' is defined as

$$\frac{\partial \vec{r}}{\partial x} = \lim_{\delta t \to 0} \left[\frac{\vec{r}(x+\delta x, y, z) - \vec{r}(x,y,z)}{\delta x}\right]$$

provided the limit exists

similarly $$\frac{\partial \vec{r}}{\partial y} = \lim_{\delta t \to} \left[\frac{\vec{r}(x,y+\delta y, z) - f(x,y,z)}{\delta y}\right]$$

and $$\frac{\partial \vec{r}}{\partial z} = \lim_{\delta t \to 0} \left[\frac{\vec{r}(x,y,z+\delta z) - \vec{r}(x,y,z)}{\delta z}\right]$$

provided limit exists

③ Derivative of a Constant Vector

Def:- A vector $\vec{a}$ is said to be constant vector when its magnitude and direction not change. The its derivative will be zero

$$\frac{d\vec{a}}{dt} = 0$$

General rule of Differentiation

④ Theorem:- (Derivative of sum. If $\vec{u}(t)$ and $\vec{v}(t)$ be two differentiable function of the scalar t to show that $\frac{d}{dt}(\vec{u} \pm \vec{v}) = \frac{d\vec{u}}{dt} \pm \frac{d\vec{v}}{dt}$

Proof:- Let $\vec{w} = \vec{u} \pm \vec{v}$ —①

Let $\delta\vec{u}$ and $\delta\vec{v}$ be the small increments in $\vec{u}$ and $\vec{v}$ respectively and $\delta\vec{w}$ be corresponding increment in $\vec{w}$ Then $\vec{w}+\delta\vec{w} = (\vec{u}+\delta\vec{u}) \pm (\vec{v}+\delta\vec{v})$ ②

Subtracting the corresponding small increment in $\vec{w}$

Then $\vec{w}+\delta\vec{w}-\vec{w} = (\vec{u}+\delta\vec{u}) \pm (\vec{v}+\delta\vec{v}) - (\vec{u} \pm \vec{v})$

or $\delta\vec{w} = \delta\vec{u} \pm \delta\vec{v}$

dividing both side by δt we get

$$\frac{\delta\vec{w}}{\delta t} = \frac{\delta\vec{u}}{\delta t} \pm \frac{\delta\vec{v}}{\delta t}$$

Now, $\lim_{\delta t \to 0} \frac{\delta\vec{w}}{\delta t} = \lim_{\delta t \to 0} \frac{\delta\vec{u}}{\delta t} + \lim_{\delta t \to 0} \frac{\delta\vec{v}}{\delta t}$

or $\frac{d\vec{w}}{dt} = \frac{d\vec{u}}{dt} \pm \frac{d\vec{v}}{dt}$

Hence $\frac{d}{dt}(\vec{u} \pm \vec{v}) = \frac{d\vec{u}}{dt} \pm \frac{d\vec{v}}{dt}$

Theorem :- (Derivative of vector product of two vectors function

If $\vec{u}(t)$ and $\vec{v}(t)$ be two differential functions of the scalar t to show that

$$\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u} \times \frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt} \times \vec{v}$$

proof :— $\vec{w} = \vec{u} \times \vec{v}$

Let $\delta\vec{u}$ and $\delta\vec{v}$ be the small increments in $\vec{u}$ and $\vec{v}$ respectively and $\delta\vec{w}$ be the corresponding small increment in $\vec{w}$

Then $\vec{w} + \delta\vec{w} = (\vec{u} + \delta\vec{u}) \times (\vec{v} + \delta\vec{v})$

$\therefore \vec{w} + \delta\vec{w} = \vec{u} \times \vec{v} + \vec{u} \times \delta\vec{u} + \delta\vec{u} \times \vec{v} + \delta\vec{u} \times \delta\vec{v}$ — (2)

Subtracting the corresponding side of (1) from (2) we get

$\vec{w} + \delta\vec{w} - \vec{w} = \vec{u} \times \vec{v} + \vec{u} \times \delta\vec{v} + \delta\vec{u} \times \vec{v} + \delta\vec{u} \times \delta\vec{v} - \vec{u} \times \vec{v}$

or $\delta\vec{w} = \vec{u} \times \delta\vec{v} + \delta\vec{u} \times \vec{v} + \delta\vec{u} \times \delta\vec{v}$

Dividing both side by δt we get

$$\frac{\delta\vec{w}}{\delta t} = \vec{u} \times \frac{\delta\vec{v}}{\delta t} + \frac{\delta\vec{u}}{\delta t} \times \vec{v} + \frac{\delta\vec{u}}{\delta t} \times \frac{\delta\vec{v}}{\delta t} \cdot \delta t$$

Now proceeding to limits as $\delta t \to 0$ we get

$$\lim_{\delta t\to 0}\frac{\delta \vec{w}}{\delta t} = \vec{u}\times \lim_{\delta t\to 0}\frac{\delta \vec{v}}{\delta t} + \lim_{\delta t\to 0}\frac{\delta \vec{u}}{\delta t}\times \vec{v}$$
$$+\lim_{\delta t\to 0}\frac{\delta \vec{u}}{\delta t}\times \lim_{\delta t\to 0}\frac{\delta \vec{v}}{\delta t}\cdot \lim_{\delta t\to 0}\delta t$$

or $$\frac{d\vec{\omega}}{dt} = \vec{u}\times\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\times\vec{v} + \frac{d\vec{u}}{dt}\times\frac{d\vec{v}}{dt}\cdot 0$$

$$\text{or } \frac{d}{dt}(\vec{u}\times\vec{v}) = \vec{u}\times\frac{d\vec{v}}{dt} + \frac{d\vec{u}}{dt}\times\vec{v}$$

problem ① If $\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$

Show that $\vec{r}\times\frac{d\vec{r}}{dt} = \omega\,\vec{a}\times\vec{b}$

Soln :- From the given relation

$$\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$$

we have on differentiation

$$\frac{d\vec{r}}{dt} = -a\omega\sin\omega t + \vec{b}\,\omega\cos\omega t$$

$$\therefore \vec{r}\times\frac{d\vec{r}}{dt} = (\vec{a}\cos\omega t + \vec{b}\sin\omega t)\times(-\vec{a}\,\omega\sin\omega t + \vec{b}\,\omega\cos\omega t)$$

$$= \vec{a}\times\vec{b}\,\omega\cos^2\omega t - \vec{b}\times\vec{a}\,\omega\sin^2\omega t$$
$$= \vec{a}\times\vec{b}\,\omega\cos^2\omega t + \vec{a}\times\vec{b}\,\omega\sin^2\omega t$$
$$= \omega\,\vec{a}\times\vec{b}\,(\sin\omega^2\omega t + \cos^2\omega t)$$
$$= \omega\,\vec{a}\times\vec{b}$$

problem ② Show that the necessary and sufficient condition for the vector function $\vec{u}$ of a scalar variable t to have constant magnitude is

$\vec{u} \cdot \frac{d\vec{u}}{dt} = 0$ ie. $\vec{u} \perp \frac{d\vec{u}}{dt}$

proof :— ~~Let magnitude of $\vec{u}$ be u (const)~~ Necessary condition

Let the magnitude of $\vec{u}$ be u which is constant

if u be constant then $\frac{du}{dt} = 0$

Now $\vec{u} \cdot \vec{u} = u^2$

or $\frac{d}{dt}(\vec{u} \cdot \vec{u}) = 0$

or $\vec{u} \frac{d\vec{u}}{dt} + \frac{d\vec{u}}{dt} \cdot \vec{u} = 0$

or $2\vec{u} \frac{d\vec{u}}{dt} = 0$

or $\vec{u} \cdot \frac{d\vec{u}}{dt} = 0$ i.e $\vec{u} \perp \frac{d\vec{u}}{dt}$

This proves the necessary condition

Sufficient condition :-

we have $\vec{u} \cdot \frac{d\vec{u}}{dt} = 0$

or $2\vec{u} \cdot \frac{d\vec{u}}{dt} = 0$

or $\vec{u} \cdot \frac{d\vec{u}}{dt} + \frac{d\vec{u}}{dt} \cdot \vec{u} = 0$

or $\frac{d}{dt}(\vec{u} \cdot \vec{u}) = 0$

or $\frac{d}{dt}(u^2) = 0$

This shows that the magnitude u of vector $\vec{u}$ is constant

Hence the sufficient condition is proved.